FINITENESS PROPERTIES AND PROFINITE COMPLETIONS

ALEXANDER LUBOTZKY

Einstein Institute of Mathematics
Hebrew University of Jerusalem
Jerusalem 91904, Israel
alex.lubotzky@mail.huji.ac.il

Abstract. In this note we show that various (geometric/homological) finiteness properties are not profinite properties. For example for every $1 \leq k, \ell \leq \mathbb{N}$, there exist two finitely generated residually finite groups Γ_1 and Γ_2 with isomorphic profinite completions, such that Γ_1 is strictly of type F_k and Γ_2 of type F_ℓ.

1. Introduction

Let Γ be a finitely generated group. What properties of Γ can be deduced from its finite quotients? The question makes sense only for residually finite groups. Moreover, two finitely generated groups Γ_1 and Γ_2 have the same set of (isomorphism classes of) finite quotients if and only if they have isomorphic profinite completion $\hat{\Gamma}_1 \simeq \hat{\Gamma}_2$ (cf. [DFPR]). Let us therefore define:

Definition 1.1. A property of groups $\boldsymbol{P}$ is called a **profinite property** if whenever Γ_1 and Γ_2 are finitely generated residually finite groups with $\hat{\Gamma}_1 \simeq \hat{\Gamma}_2$ and Γ_1 has property $\boldsymbol{P}$, so does Γ_2.

In recent years there has been a growing interest in understanding what properties are profinite properties (cf. [B1], [La], [Ak] and the references therein and especially [GZ] for a historical review and a systematic program of research). This resembles the quite intensive area of study of "geometric properties", i.e. properties shared by all pairs of finitely generated groups with quasi-isometric Cayley graphs.

The current note was sparked by a lecture given by Martin Bridson in Park-City in July 2012, where he presented an example of two finitely generated residually finite groups Γ_1 and Γ_2 with isomorphic profinite completions, such that Γ_2 is finitely presented while Γ_1 is not. So, in

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the above terminology, the property of being finitely presented is not a profinite property. See [BR] for this and more.

Our main result is:

Theorem 1.2. *For every r and s there exists finitely generated residually finite groups Γ_r and Γ_s such that Γ_r has property F_r (and not F_{r+1}), Γ_s has F_s (and not F_{s+1}) and $\hat{\Gamma}_r$ is isomorphic to $\hat{\Gamma}_s$.*

Recall that a countable group Γ is said to have property F_m if there exists an Eilenberg-MacLane complex $K(\Gamma, 1)$ with finite m-skeleton. Every group is of type F_0. Property F_1 is equivalent to being finitely generated while property F_2 is equivalent to being finitely presented. Our theorem is therefore a far-reaching generalization of Bridson-Reid's example.

Let us denote $\phi(\Gamma) = sup\{m | \Gamma \text{ has property } F_m\}$ and call $\phi(\Gamma)$-the **finiteness length** of Γ. So our theorem gives:

Corollary 1.3. The finiteness length is not a profinite property.

The theorem is deduced in §2 from various deep results on arithmetic groups over positive characteristic function fields. A similar trick is used to deduce the following somewhat surprising result:

Proposition 1.4. *Being residually solvable (resp. residually nilpotent, residually $-p$) is not a profinite property.*

The same trick when applied in §3 for arithmetic groups in characteristic zero gives us

Proposition 1.5. (a) *Being torsion-free is not a profinite property.*
(b) *Having trivial center is not a profinite property.*

Additional results on lattices in Lie groups give:

Proposition 1.6. *Cohomological dimension is not a profinite property*

We conclude in §4 with some related remarks, questions and suggestions for further research.

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2. Arithmetic groups of positive characteristic

We now prove a much stronger version of Theorem 1.2.

Theorem 2.1. *For every $1 \leq n \in \mathbb{N}$, there exist residually finite groups $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ with isomorphic profinite completions and with $\phi(\Gamma_i) = i$ for $0 \leq i \leq n$.*

The proof relies on some properties of positive characteristic arithmetic groups. Some remarkable results have been proven recently on the finiteness properties of these groups (cf. [BW1] [BW2]) but for our case the more classical results on SL_2 suffice. So, let us formulate them in a way ready for us to use:

Theorem 2.2 (Stuhler [St]). *Fix a prime power q, and $\mathcal{O} = \mathbb{F}_q[t]$. Let S be a set of irreducible polynomials in $\mathbb{F}_q[t]$ and*

$$\mathcal{O}_S = \left\{ \frac{f(x)}{g(x)} \in \mathbb{F}_q(t) \;\middle|\; \begin{array}{c} f(x), g(x) \in \mathbb{F}_q[t] \text{ and} \\ \text{the only irreducible} \\ \text{divisors of } g(x) \text{are from } S \end{array} \right\}.$$

Then $\phi(\mathrm{SL}_2(\mathcal{O}_S)) = |S|$.

So, for $S = \emptyset, \mathrm{SL}_2(\mathcal{O}_S) = \mathrm{SL}_2(\mathbb{F}_q[t])$ is of type F_0 but not F_1, i.e. not finitely generated, while if $|S| = 1$, $\mathrm{SL}_2(\mathcal{O}_S)$ is finitely generated (F_1) but not finitely presented. On the other hand, if $|S| \geq 2$, $\mathrm{SL}_2(\mathcal{O}_S)$ is always finitely presented.

Theorem 2.3 (Serre [Se]). *If $|S| \geq 1$, then $\mathrm{SL}_2(\mathcal{O}_S)$ has the congruence subgroup property, i.e.*

$$\widehat{\mathrm{SL}_2(\mathcal{O}_S)} = \mathrm{SL}_2(\hat{\mathcal{O}}_S).$$

Let us spell out the concrete meaning of the last result: $\hat{\mathcal{O}}_S$-the profinite completion of $\mathcal{O}_S$ is equal to: $\prod\limits_{p \notin S} (\mathcal{O}_S)_{\hat{p}}$ where $(\mathcal{O}_S)_{\hat{p}}$ is the completion of $\mathcal{O}_S$ with respect to the topology of $\mathcal{O}_S$ determined by the ideal (p) generated by the irreducible polynomial p (and its powers). It is easy to see that $(\mathcal{O}_S)_{\hat{p}} \simeq \mathcal{O}_{\hat{p}}$.

Remark 2.4. If S is an infinite set of irreducible polynomials, and $\mathcal{O}_S$ is defined in the same way as for a finite set, then Theorem 2.3 is still valid for $\mathrm{SL}_2(\mathcal{O}_S)$. This follows either from the proof in [Se] or from the fact that such $\mathrm{SL}_2(\mathcal{O}_S)$ is the union of $\mathrm{SL}_2(\mathcal{O}_{S'})$ where S' runs over the finite subsets of S. Note also that as long as S is not the set of **all** irreducible polynomials in $\mathbb{F}_q[x]$, $\mathrm{SL}_2(\mathcal{O}_S)$ is residually finite since $\mathrm{SL}_2(\mathcal{O}_S) \hookrightarrow \mathrm{SL}_2((\mathcal{O}_S)_{\hat{p}})$, for every $p \notin S$, and the latter group is a profinite group. Of course, if S is the set of all irreducible polynomials then $\mathcal{O}_S = \mathbb{F}_q(t)$ and $\mathrm{SL}_2(\mathbb{F}_q(t))$ has no finite index subgroup (in fact, it is simple mod its center).

Before moving on to the proof of the theorem, we need the following lemma which is surely well known to experts. We were not able to allocate an explicit reference. We are grateful to Shmuel Weinberger who showed us how to deduce it quickly from [Wa].

Lemma 2.5. *Let Γ_1 and Γ_2 be two countable groups. Then $\phi(\Gamma_1 \times \Gamma_2) = \min(\phi(\Gamma_1), \phi(\Gamma_2))$.*

Proof. By a result of Wall [Wa, Theorem A, p. 58] it is equivalent for a homotopy type to have property F_m or to be dominated by a space of type F_m. Thus, if $\Gamma_1 \times \Gamma_2$ has type F_m so do Γ_1 and Γ_2, i.e. $\phi(\Gamma_i) \geq \phi(\Gamma_1 \times \Gamma_2)$. On the other hand, if both $K(\Gamma_1, 1)$ and $K(\Gamma_2, 1)$ have finite m-skeleton, so does $K(\Gamma_1 \times \Gamma_2, 1) = K(\Gamma_1, 1) \times K(\Gamma_2, 1)$ and hence $\phi(\Gamma_1 \times \Gamma_2) \geq \min\{\phi(\Gamma_i) | i = 1, 2\}$ and the Lemma is proven. □

We are now ready to prove Theorem 2.1: Fix $1 \leq n \in \mathbb{N}$ and fix a set of n irreducible polynomials $S = \{p_1, \dots, p_n\}$ in $\mathbb{F}_q[x]$. Denote for $i = 1, \dots, n$, $S_i = \{p_1, \dots, p_i\}$. Now choose for some $m \geq n$, a set T of m irreducible polynomials with $T \cap S = \emptyset$. For $i = 1, \dots, n$, denote $T_i = T \cup \{p_{i+1,\dots,}p_n\}$, so $T_n = T$ and $S_i \cup T_i = S \cup T$. Finally let $\Gamma_i = \mathrm{SL}_2(\mathcal{O}_{S_i}) \times \mathrm{SL}_2(\mathcal{O}_{T_i})$.

We claim: (a) $\phi(\Gamma_i) = i$. Indeed by Theorem 2.2, $\phi(\mathrm{SL}_2(\mathcal{O}_{S_i})) = i$ while $\phi(\mathrm{SL}_2(\mathcal{O}_{T_i})) = m + n - i$. Hence, by Lemma 2.5, $\phi(\Gamma_i) = \min(i, m + n - i) = i$.

(b) $\hat{\Gamma}_1 \simeq \hat{\Gamma}_2 \simeq \cdots \simeq \hat{\Gamma}_n$. In fact, as $S_i \cup T_i = S \cup T$, we have by Theorem 2.3 and the explanation following it:

$$\begin{aligned}
\hat{\Gamma}_i &= \prod_{p \notin S_i} \mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \times \prod_{p \notin T_i} \mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \\
&\cong \prod_{p \notin T \cup S} (\mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \times \mathrm{SL}_2(\mathcal{O}_{\hat{p}})) \times \prod_{p \in T \cup S} \mathrm{SL}_2(\mathcal{O}_p).
\end{aligned}$$

This shows that the isomorphism type of the profinite completion is independent of i.

We still have to show that there exists a countable group Γ_0 which is residually finite and not finitely generated (so $\phi(\Gamma_0) = 0$) and with the same profinite completion as of $\Gamma_1, \dots, \Gamma_n$. To this end let R be an infinite set of irreducible polynomials in $\mathbb{F}_q[t]$, whose complement $\bar{R}$ is nonempty. In this case $\mathrm{SL}_2(\mathcal{O}_R)$ is an ascending union of finitely generated groups and hence not finitely generated. Still by Remark 2.4, Serre's Theorem applies and both $\mathrm{SL}_2(\mathcal{O}_R)$ and $\mathrm{SL}_2(\mathcal{O}_{\bar{R}})$ have the

congruence subgroup property. Hence

$$\widehat{\mathrm{SL}_2(\mathcal{O}_R) \times \mathrm{SL}_2}(\mathcal{O}_{\bar{R}}) = \mathrm{SL}_2(\hat{\mathcal{O}}_R) \times \mathrm{SL}_2(\hat{\mathcal{O}}_{\bar{R}}) = \prod_p \mathrm{SL}_2(\mathcal{O}_{\hat{p}})$$

where this time p runs exactly once over all the irreducible polynomials in $\mathbb{F}_q[t]$.

Let us now take $\Gamma_0 = \mathrm{SL}_2(\mathcal{O}_{T\cup S}) \times \mathrm{SL}_2(\mathcal{O}_R) \times \mathrm{SL}_2(\mathcal{O}_{\bar{R}})$. This is not a finitely generated group and its profinite completion is

$$\begin{aligned} &\prod_{p \notin T\cup S} \mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \times \prod_{\text{all } p} \mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \\ &= \prod_{p \notin T\cup S} (\mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \times \mathrm{SL}_2(\mathcal{O}_{\hat{p}})) \times \prod_{p \in T\cup S} \mathrm{SL}_2(\mathcal{O}_{\hat{p}}). \end{aligned}$$

So, $\hat{\Gamma}_0$ is isomorphic to $\hat{\Gamma}_i$ for all $i = 1, \ldots, n$ and Theorem 2.1 is proven.

Proof of Proposition 1.4. Let us first say that by residually solvable (resp. nilpotent, p) we mean here that the homomorphisms to **finite** solvable (resp. nilpotent, p) groups separate the points of the group. But, actually the proof will work also in the other sense, i.e. when no finite assumption is made. Anyway Proposition 1.4 is somewhat surprising since it shows that there are residually finite groups Γ_1 and Γ_2 with the same finite quotients, and in particular, the same finite solvable quotients; still, for Γ_1, the finite solvable quotients separate its points, while for Γ_2 they do not.

For the proof we will use again the notations used in the proof of Theorem 2.1. For simplicity, assume $q \geq 4$. Let $S_1 = \{p_1, p_2\}$ be a set of two primes in $\mathbb{F}_q[x]$. Write Γ_S for $\mathrm{SL}_2(\mathcal{O}_S)$ and for a prime p, $p \notin S$, $\Gamma_S(p) = \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_S) \to \mathrm{SL}_2(\mathcal{O}_S/(p)))$, the congruence subgroup $\mod p$.

Let now p_3, p_4 be two different primes not in S. Denote

$$\begin{aligned} \Gamma_1 &= \Gamma_S(p_3) \times \Gamma_S(p_4) \\ \text{and } \Gamma_2 &= \Gamma_S \times \Gamma_S(p_3 p_4), \\ \text{where } \Gamma_S(p_3 p_4) &= \Gamma_S(p_3) \cap \Gamma_S(p_4) \end{aligned}$$

is the congruence subgroup $\mod p_3 p_4$. Now, Γ_1 is a finitely presented residually $-p$ group. Indeed $\Gamma_S(p_3)$ is a subgroup of its closure in $\mathrm{SL}_2(\mathcal{O}_S)_{\hat{p}_3})$, i.e. SL_2 over the p_3-closure of $\mathcal{O}_S$. But it is inside $\mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_S)_{\hat{p}_3} \to \mathrm{SL}_2((\mathcal{O}_S)_{\hat{p}_3}/(p_3)))$ - the p_3-congruence subgroup which is a pro-p group (this time p is the rational prime such that $q = p^e$ for some e). Hence $\Gamma_S(p_3)$ is residually-p group. The same holds for $\Gamma_S(p_4)$. On the other hand, Γ_S has no non-trivial solvable quotient.

Indeed, by a well known result of Margulis, every normal subgroup of Γ_S is either finite or of finite index, [Ma, Chap. VIII, Theorm (2.6), p. 265]. Moreover, from the affirmative solution of the congruence subgroup problem we deduce that every non-trivial finite quotient of Γ_S is mapped onto $\mathrm{PSL}_2(q^a)$ for some $a \geq 1$. As $q \geq 4$, these are non-abelian finite simple groups. This implies that Γ_2 is not residually solvable.

Finally, by a similar argument as in the proof of Theorem 2.1, we see that

$$\begin{aligned}
\hat{\Gamma}_1 =& \hat{\Gamma}_S(p_3) \times \hat{\Gamma}_S(p_4) = \prod_{p \neq p_3} \mathrm{SL}_2(\mathcal{O}_{\hat{p}}) \times \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_{\hat{p}_3}) \to \mathrm{SL}_2(\mathcal{O}_{\hat{p}_3}/(p_3)) \\
&\times \prod_{p \neq p_4} \mathrm{SL}_2(\mathcal{O}_p) \times \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_{\hat{p}_4}) \to \mathrm{SL}_2(\mathcal{O}_{\hat{p}_4}/(p_4)) \\
&\cong \prod_{p \neq p_3, p_4} \mathrm{SL}_2(\mathcal{O}_{\hat{p}})^2 \times \mathrm{SL}_2(\mathcal{O}_{\hat{p}_3}) \times \mathrm{SL}_2(\mathcal{O}_{\hat{p}_4}) \\
&\times \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_{\hat{p}_3}) \to (\mathrm{SL}_2(\mathcal{O}_{\hat{p}3}/(p_3))) \times \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_{\hat{p}_4}) \to \mathrm{SL}_2(\mathcal{O}_{\hat{p}_4}/(p_4))).
\end{aligned}$$

While

$$\begin{aligned}
\hat{\Gamma}_2 =& \mathrm{S}\widehat{\mathrm{L}_2(\mathcal{O}_S)} \times \Gamma\widehat{_S(p_3p_4)} \\
=& \prod_p \mathrm{SL}_2(\mathcal{O}_p) \times \prod_{p \neq p_3, p_4} \mathrm{SL}_2(\mathcal{O}_p) \\
&\times \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_{\hat{p}_3} \to \mathrm{SL}_2(\ \mathcal{O}_{\hat{p}_3}/(p_3)) \\
&\times \mathrm{Ker}(\mathrm{SL}_2(\mathcal{O}_{\hat{p}_4}) \to \mathrm{SL}_2(\mathcal{O}_{p_4}/(p_4))
\end{aligned}$$

and therefore

$$\hat{\Gamma}_2 \simeq \hat{\Gamma}_1.$$

Proposition 1.4 is now proven since Γ_1 is residually finite-p, while Γ_2 is not even residually solvable. $\square$

3. Arithmetic groups of zero charactistic

Let us start with proving Proposition 1.5:

This time let $\Gamma_S = \mathrm{SL}_2(\mathbb{Z}_S)$ where S is a finite set of rational primes and

$$\mathbb{Z}_S = \left\{ \frac{a}{b} \,\middle|\, a, b \in \mathbb{Z} \text{ and all prime divisors of } b \text{ are in } S \right\}.$$

As Γ_S contains $\mathrm{SL}_2(\mathbb{Z})$, it has nontrivial torsion. But for every $2 \neq \ell \in \mathbb{Z}$ which is not in S, the congruence subgroup

$$\Gamma_S(\ell) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_S) \to \mathrm{SL}_2(\mathbb{Z}_S/\ell\mathbb{Z}_S))$$

is torsion free.

By a result of Serre [Se] $\mathrm{SL}_2(\mathbb{Z}_S)$ has the congruence subgroup property whenever $S \neq \emptyset$. This means that $\widehat{\mathrm{SL}_2(\mathbb{Z}_S)} \simeq \mathrm{SL}_2(\hat{\mathbb{Z}}_S) = \prod_{p \notin S} \mathrm{SL}_2(\mathbb{Z}_p)$ when $\mathbb{Z}_p$ is the ring of p-adic integers. Now, for $\Gamma_S(\ell)$ we have (still assuming $S \neq \emptyset$):

$$\widehat{\Gamma_S(\ell)} \cong \Big(\prod_{\substack{p \notin S \\ p \nmid \ell}} \mathrm{SL}_2(\mathbb{Z}_p)\Big) \times \prod_{\substack{p \notin S \\ p \mid \ell}} \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{Z}_p/\ell\mathbb{Z}_p)$$

Now let

$S = \{7\}$, $\ell = 15 = 3 \cdot 5$, $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z}_S) \times \Gamma_S(\ell)$ and $\Gamma_2 = \Gamma_S(3) \times \Gamma_S(5)$.

Clearly Γ_1 has torsion while Γ_2 does not. Moreover Γ_1 has nontrivial center, while Γ_2 is centerless. Still

$$\hat{\Gamma}_1 = \Big(\prod_{p \neq 7} \mathrm{SL}_2(\mathbb{Z}_p)\Big) \times \Big(\prod_{p \neq 7,3,5} \mathrm{SL}_2(\mathbb{Z}_p)\Big) \\ \times \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_3) \to \mathrm{SL}_2(\mathbb{F}_3)) \times \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_5) \to \mathrm{SL}_2(\mathbb{F}_5))$$

while

$$\hat{\Gamma}_2 = \Big(\prod_{p \neq 3,7} \mathrm{SL}_2(\mathbb{Z}_p)\Big) \times \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_3) \to \mathrm{SL}_2(\mathbb{F}_3)) \\ \times \Big(\prod_{p \neq 5,7} \mathrm{SL}_2(\mathbb{Z}_p)\Big) \times \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_5) \to \mathrm{SL}_2(\mathbb{F}_5)).$$

So both groups $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are isomorphic to:

$$\prod_{p \neq 3,5,7} \big(\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)\big) \times \mathrm{SL}_2(\mathbb{Z}_3) \times \mathrm{SL}_2(\mathbb{Z}_5) \\ \times \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z}_3) \to \mathrm{SL}_2(\mathbb{F}_3)) \times (\mathrm{Ker}\mathrm{SL}_2(\mathbb{Z}_5) \to \mathrm{SL}_2(\mathbb{F}_5)).$$

This proves the proposition.

Proof of Proposition 1.6. The efforts to answer the Grothendieck problem (cf. [Gr],[PT],[BL],[Py],[BG]) have led to a number of methods and results of the following kind: There exist finitely generated residually finite groups Γ_1 and Γ_2 with an injective map $\varphi : \Gamma_1 \to \Gamma_2$, such that the induced map $\hat{\varphi} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism while Γ_1 is not isomorphic to Γ_2.

Let us recall the construction from [BL]: there $\Gamma_2 = L \times L$ when L is a cocompact torsion free lattice in $G = Sp(n, 1)$ or $G = F_4^{(-20)}$. In particular the cohomological dimension $cd(\Gamma_2) = 2cd(L)$ and $cd(L) = \dim(G/K)$ when K is a maximal compact subgroup of G. On the other hand Γ_1 is obtained as follows: Let $\pi : L \to M$ be an infinite finitely presented quotient of L such that M has no finite index subgroup.

(Such a quotient M exists by [Ol] as L is hyperbolic group.) Let Γ_1 be the fiber product over π, i.e. $\Gamma_1 = \{(a,b) \in L \times L \big| \pi(a) = \pi(b)\}$. Then Γ_1 is of infinite index, so $cd(\Gamma_1) \lneqq cd(\Gamma_2) = 2cd(L)$ containing the diagonal subgroup (and so $cd(\Gamma_1) \geq cd(L)$). It is shown in [BL]that $\hat{\Gamma}_1 \simeq \hat{\Gamma}_2$ and hence Proposition 1.6 follows.

Let us remark, that our result here is not as strong as Theorem 1.2. We do not know to give, for arbitrary r and s in $\mathbb{N}$, examples of Γ_1 and Γ_2 with $cd(\Gamma_1) = r$, $cd(\Gamma_2) = s$ and $\hat{\Gamma}_1 \simeq \hat{\Gamma}_2$. This is probably a difficult problem: recall that $cd(\Gamma) = 1$ if and only if Γ is a free group. It is a long-standing open problem (usually attributed to Remeslenikov, cf.[GZ]) whether freeness is a profinite property.

4. Remarks and problems

We have discussed throughout the paper only countable groups and especially finitely generated groups (which is the most interesting case for our problem) but one can also say something about uncountable groups:

By the recent remarkable result of Nikolov and Segal [NS], every finite index subgroup of a finitely generated profinite group G is open. This means that $\hat{G} = G$. Applying this for $G = \hat{\Gamma}$ the profinite completion of a finitely generated discrete group Γ, we deduce that $\hat{\Gamma} \simeq \hat{G}$. Hence for every finitely generated infinite, residually finite discrete group Γ, we deduce that $\hat{\Gamma} \simeq \hat{G}$. Hence for every finitely generated discrete group Γ there exists an uncountable group G with $\hat{\Gamma} = \hat{G}$. On the other hand we do not know if the following is true: For every finitely generated residually finite group Γ there exists a countable **non** finitely generated (but residually finite) group G, with $\hat{\Gamma} = \hat{G}$.

In light of Remeslenikov's problem mentioned in the previous section, this would be an interesting problem to know if such G exists when Γ is a finitely generated free group.

The main interest of [BR] is with pairs of groups Γ_1 and Γ_2 which have the same "nilpotent genus" (i.e. for every $m \in \mathbb{N}$, $\Gamma_1/\Gamma_1^{(m)} \simeq \Gamma_2/\Gamma_2^{(m)}$ where $\Gamma_i^{(m)}$ is the m term of the lower central series of Γ_i.) This implies (though not equivalent) to having the same pro(finite) nilpotent completion. Of course, if the profinite completions are isomorphic so are the pronilpotent completions. But the examples we presented in the proofs of the results of this paper are usually not residually nilpotent. This can be fixed quite easily by switching each time to a suitable congruence subgroup as we did in the proof of Proposition 1.4

(and recalling that proper principal congruence subgroups are always residually nilpotent). We leave the details to the reader. This is usually easy except in the proof of Proposition 1.5, where some care has to be taken: Here it is important to replace Γ_1 and Γ_2 by their mod 2 congruence subgroups, as we want that the new Γ_1 still has torsion and center. (This is the reason we presented the proof there with $S = \{7\}$ and not with $S = \{2\}$. Of course for proving the original Proposition 1.5, we could use $S = \{2\}$ or $S = \{p\}$ for any prime $p \neq 3, 5$. It is only for the pronilpotent version that it is important not to use $S = \{2\}$ as for this case Γ_S has **no** mod 2 congruence subgroup.)

There is however an interesting difference between our residually nilpotent groups and the ones presented in [BR]. There, all examples are residually torsion free nilpotent. But our examples are never such. The examples Γ_i presented for Theorems 1.2 (or 2.1) and Propositions 1.4 and 1.5 have the property (FAb) i.e., for every finite index subgroup $\triangle$, $\triangle/[\triangle, \triangle]$ is finite, since they have the congruence subgroup property. The same holds for Γ_2 of the proof of Proposition 1.6, since Γ_2 has Kazhdan property T. (It follows that Γ_1 also has (FAb), since (FAb) is clearly a profinite property). For all our examples, when Γ_1 and Γ_2 have the same pronilpotent completion they also have the same nilpotent genus.

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